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ABSTRACT

Shock relations are established for the dense high temperature gas, ($\rho \sim 10^9$ gr/c.c. , $T \sim 10^9$ °K) , introducing relativistic concepts. It is assumed that the self-consistent static electric field does not play a significant role in the shock relations.

From these relations, the heating effect due to the passage of the shock waves through a degenerate gas is deduced explicitly, indicating large heating effects. The results may be of significance in understanding the supernova phenomenon.

I. INTRODUCTION

The Rankine-Hugoniot relations for shock waves are the conservation relations for mass, momentum and energy of the flow in a continuous medium. These have long been known for the case of the ideal polytropic gas.(1) P. Lal and P. L. Bhatnagar(2) have indicated one method for extending these relations to the degenerate Fermi-Dirac gas. In this paper, we shall rederive these relations for the F-D gas; we find that our results differ from those of Lal and Bhatnagar, because the basic physical picture on which the derivation is based is itself different.

Due to the extremely high mass density in the interior of a white dwarf or a presupernova star ($\rho \sim 10^6 \sim 10^9$ gr./c.c.), the matter is completely ionized and highly degenerate even at temperatures of the order of 10^9 °K. The electron gas plays a dominant role, and it has been shown by Landau(3) that an ordinary sound wave cannot propagate in a neutral degenerative medium unless the temperature (in energy units) is comparable to the Fermi energy. On the other hand, Landau has shown that in the presence of interaction, a new propagation mode, termed zero sound, can exist.

It is our purpose to establish the shock relations for the dense degenerate gas and to extract some of the physical features a shock wave must have in such a plasma. We do not consider here how such a shock wave may be formed, but assuming that shock waves of the zero-sound type exist, we investigate their properties.

We make the following approximation. The self-consistent static electric field is not included in the distribution function explicitly. We expect that the electrostatic energies in the front and rear regions of the shock wave are of the same order of magnitude so that the contributions to the enthalpy due to interaction energies will tend to cancel out. This point should be studied and clarified in the future.

The phase velocity of the zero-sound is of the order of the velocity of light, for the temperatures and densities considered, and this property, together with the conservation of the energy, gives a large heating effect, perhaps as large as a factor of 20 in temperature. This heating process can occur almost instantaneously due to the large propagation velocity of the zero-sound.

Since the existence of zero-sound depends on an interparticle interaction, we have examined the question of whether such an interaction alters the form of the Rankine-Hugoniot relations. This study is presented in the Appendix, where it is concluded that the form of these relations is unchanged.

II.

We shall assume extreme conditions of density and temperature, corresponding to a presupernova core. We assume

$$\left.\begin{array}{lll} \rho \sim 10^9 \text{ gr./c.c.} & & \text{density of mass} \\ T \sim 10^9 \, {}^\circ\text{K.} & & \text{Temperature} \end{array}\right\} \qquad (1)$$

Under these circumstances, the gas is completely ionized, but the electron gas is strongly degenerate. The ion distribution is principally determined by the requirement of electrical neutrality.

The conditions in (1) can be expressed most conveniently in terms of new variables:

$$x \equiv \frac{p_0}{mc} \qquad (p_0 = \text{Fermi momentum})$$

$$\varepsilon \equiv \frac{KT}{mc^2} \qquad (K = \text{Boltzmann constant})$$

and

$$\mu_0 \equiv \frac{E_F}{KT} \qquad (E_F = \text{Fermi energy}).$$

These parameters have the following numerical values:

$$x = 8, \quad \varepsilon = \frac{1}{6} \quad \text{and} \quad \mu_0 = 45 \qquad (2)$$

The symbols m and c have their usual meanings: mc^2 is the rest energy of the electron.

Following S. Chandrasekhar(4), one finds for the free electron gas the pressure P, number N_e, and internal energy U_e:

$$P = \frac{\pi m^4 c^5}{3 h^3} f(x) \left[1 + 4\pi^2 \left(\frac{KT}{mc^2}\right)^2 \frac{x (x^2+1)^{\frac{1}{2}}}{f(x)} + \cdots \right] \qquad (3)$$

$$N_e = \frac{8\pi V m^3 c^3}{3 h^3} x^3 \left[1 + \pi^2 \left(\frac{KT}{mc^2}\right)^2 \frac{2x^2+1}{x^4} + \cdots \right] \qquad (4)$$

$$U_e = \frac{\pi V m^4 c^5}{3 h^3} g(x) \left[1 + 4\pi^2 \left(\frac{KT}{mc^2}\right)^2 \frac{(3x^2+1)(x^2+1)^{\frac{1}{2}} - (2x^2+1)}{x \; g(x)} + \cdots \right] \qquad (5)$$

with

$$f(x) = x(2x^2-3)(x^2+1)^{\frac{1}{2}} + 3\sinh^{-1}x \tag{6}$$

$$g(x) = 8x^3[(x^2+1)^{\frac{1}{2}}-1] - f(x). \tag{7}$$

V refers to the total volume. $f(x)$ can be approximated as

$$f(x) = 2x^4 - 3x^2 \cdots \qquad \text{for } x \gg 1 \tag{8}$$

Compared to the above values for the electron gas, it can be shown that pressure and internal energy of the ion gas are negligible. Furthermore, the number of electrons N_e is related to the density of the gas as

$$N_e = \frac{\rho V}{\mu H}, \tag{9}$$

where μ is mean molecular weight per electron, and H is mass of the hydrogen atom. The mass density is given by

$$\rho = \mu H \frac{8\pi m^3 c^3}{3h^3}\left[x^3 + \pi^2\varepsilon^2 \frac{2x^2+1}{2x} + \cdots \right] \tag{10}$$

Having in mind that $x \approx 8$ for the situation under consideration, one finds that the internal energy of the gas per unit mass can be expressed as

$$u = \frac{p}{\rho} \frac{g(x) + \frac{4\pi^2\varepsilon^2}{x}[(3x^2+1)(x^2+1)^{\frac{1}{2}} - (2x^2+1)]}{f(x) + 4\pi^2\varepsilon^2 x(x^2+1)^{\frac{1}{2}}} \tag{11}$$

It follows that the enthalpy of the gas of unit mass, $i \equiv U + \frac{P}{\rho}$, can be expressed as

$$i = \frac{p}{\rho} \frac{8x^3[(x^2+1)^{\frac{1}{2}}-1] + \frac{4\pi^2\varepsilon^2}{x}[(4x^2+1)(x^2+1)^{\frac{1}{2}} - (2x^2+1)]}{2x^4 - 3x^2 + 4\pi^2\varepsilon^2 x(x^2+1)^{\frac{1}{2}}} + \cdots \tag{12}$$

Regarding the gas as polytropic with polytropic index γ, one finds (putting $\varepsilon = 0$)

$$\gamma \approx \frac{8x^2 - 8x + 4}{6x^2 - 8x + 7}, \qquad x \gg 1. \tag{13}$$

In the limit $x \longrightarrow \infty$, one finds γ approaches 4/3. It is known that in the

limit $x \longrightarrow 0$, for which (13) is not valid, γ approaches 5/3. Therefore, one may state that the degenerate electron gas behaves approximately as a polytropic gas for which γ varies between 4/3 and 5/3, depending on the density, and weakly on temperature.

The entropy of the gas is given as

$$S = \frac{1}{T}(U + PV - NF), \tag{14}$$

with F denoting the chemical potential. From the definition of the chemical potential

$$F = \frac{dU}{dN}, \tag{15}$$

we deduce that

$$N_{ion} F_{ion} = N_e F_{electron} = N_e E_F, \tag{16}$$

because ion and electron gas are bound to each other to satisfy the requirement of electrical neutrality. We also find

$$S_{ion} \ll S_{el},$$

so that the entropy of the gas can be represented simply by the entropy of the electron gas. This is found to be (still neglecting ε)

$$S_{el} = \frac{V\, 8\pi^3 m^2 c\, K^2 T}{3 h^3}\, x(x^2+1)^{\frac{1}{2}},$$

or

$$S \approx \frac{\pi^2 KT}{mc^2}\, \frac{K}{\mu H}\, \frac{(x^2+1)^{\frac{1}{2}}}{x^3} \qquad \text{(per unit mass)}. \tag{17}$$

One normally defines the sound velocity as

$$c^2_{sound} = \left(\frac{\partial p}{\partial \rho}\right)_{entropy}. \tag{18}$$

From eq. (17), the entropy of the gas is a function only of $x \sim \rho^{\frac{1}{3}}$. Thus,

if S is held constant, it follows that

$$\left(\frac{\partial p}{\partial \rho}\right)_{\text{entropy}} = 0 , \tag{19}$$

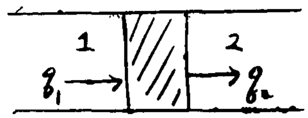
since P is a function of x only:

$$P = \frac{mc^2}{8 \mu H} \left(2x - \frac{3}{x}\right) . \tag{20}$$

Thus, we derive the fact that ordinary sound will not propagate through a degenerate gas at zero temperature. This situation was pointed out, by Landau, in discussing the zero-temperature Fermi liquid. In the presence of interaction, Landau proved that there exists another mode, called zero-sound, which propagates essentially with Fermi velocity.

III.

Let us now consider a one-dimensional shock.



Subscripts 1 and 2 refer to the front and rear regions of the shock, and q denotes velocity of the gas relative to the shock region. From Møller's[6] work, one can write the following equations expressing the conservation of number of the particles in the flow, momentum and energy in the front and rear regions of the shock wave with respect to the coordinate system to which the shock region is at rest:

$$\frac{n_1 q_1}{\sqrt{1-\frac{q_1^2}{c^2}}} = \frac{n_2 q_2}{\sqrt{1-\frac{q_2^2}{c^2}}} \equiv \Delta \qquad \text{(particle number)} \tag{21}$$

$$\frac{(i_1/c^2+1)q_1}{\sqrt{1-\frac{q_1^2}{c^2}}} = \frac{(i_2/c^2+1)q_2}{\sqrt{1-\frac{q_2^2}{c^2}}} \qquad \text{(momentum)} \tag{22}$$

$$\frac{u_1+\frac{p_1}{\rho_1}\frac{q_1^2}{c^2}}{\sqrt{1-q_1^2/c^2}} = \frac{u_2+\frac{p_2}{\rho_2}\frac{q_2^2}{c^2}}{\sqrt{1-\frac{q_2^2}{c^2}}} \qquad \text{(energy)} \tag{23}$$

Equation (23) can be rewritten as follows:

$$\frac{i_1+c^2}{\sqrt{1-q_1^2/c^2}} - \frac{p_1}{\rho_1}\sqrt{1-q_1^2/c^2} = \frac{i_2+c^2}{\sqrt{1-q_2^2/c^2}} - \frac{p_2}{\rho_2}\sqrt{1-q_2^2/c^2} \tag{23a}$$

These relations, expressing basic physical conservation laws, must of course be satisfied by the degenerate gas as well as the ordinary gas. The relativistic conception is introduced in order for this formalism to be consistent with the fact that the phase velocity of the zero-sound type shock wave is of the order of the velocity of light.

The enthalpy of the gas per unit mass, i, in eq. (12) may be written rigorously to the lowest order in ε, as

$$i = \frac{mc^2}{8\mu H} \frac{8x^3[(x^2+1)^{\frac{1}{2}}-1] + \frac{4\pi^2\varepsilon^2}{x}[(x^2+1)^{\frac{1}{2}}(4x^2+1)-(2x^2+1)]}{x^3 + \pi^2\varepsilon^2 x} \tag{24}$$

with

$$\frac{KT}{mc^2} \equiv \varepsilon ,$$

or

$$i = \frac{mc^2}{\mu H}(x - 1 + \pi^2\varepsilon^2) \tag{25}$$

From eq. (21), (22), and (25), one finds that

$$\frac{i_2/c^2 + 1}{i_1/c^2 + 1} = \frac{n_1}{n_2} ,$$

or

$$\frac{n_2 - n_1}{n_1} = \frac{(x_2 - x_1) + \pi^2(\varepsilon_2^2 - \varepsilon_1^2)}{x_1 - 1 + \pi^2\varepsilon_1^2 + \frac{\mu H}{m}} . \tag{26}$$

The right hand side can be approximated, on the assumption that x_1, ε_1 have the values given by eq. (2), as

$$\frac{n_2 - n_1}{n_1} \approx \frac{m}{\mu H}\left[(x_2 - x_1) + \pi^2(\varepsilon_2^2 - \varepsilon_1^2)\right] \tag{27}$$

We then obtain, solving for the temperatures

$$T_2^2 - T_1^2 = \left(\frac{T_1}{\varepsilon_1 \pi}\right)^2 (x_2 - x_1)\left\{\frac{\mu H}{m}\, \frac{x_2^2 + x_1 x_2 + x_1^2}{x_1^3} - 1\right\} \tag{28}$$

So long as $x_2 - x_1 > 0$, one obtains

$$T_2^2 - T_1^2 \approx T_1^2 \frac{4\mu H}{m} \frac{3(x_2 - x_1)}{x_1} . \tag{29}$$

It will be shown later that $x_2 - x_1 > 0$ is valid always for the physical process. This leads to the conclusion that

$$T_2 \approx 74\,(x_2-x_1)^{\frac{1}{2}}\,T_1\,, \tag{30}$$

with $x_2-x_1 > 0$. This implies that shock wave heats the degenerate gas and raises its temperature by a factor of 10 or more for the case with $x_2 \approx 1.02\,x_1$. It is interesting to note that a slight fractional increase in the number density can produce a large temperature rise in the gas.

We now show that

$$(x_2-x_1) > 0$$

for the physical process. From eq. (22) and (23a), one obtains

$$\frac{p_1}{\rho_1}\sqrt{1-q_1^2/c^2}\;q_1 = \frac{p_2}{\rho_2}\sqrt{1-q_2^2/c^2}\;q_2 \tag{31}$$

Making use of eq. (3) and (10), eq. (31) can be rewritten as

$$q_1\sqrt{1-q_1^2/c^2}\,(2x_1) \approx q_2\sqrt{1-q_2^2/c^2}\,\left(2x_2-\frac{3}{x_2}+\frac{4\pi^2\varepsilon_2^2}{x_2}\right), \tag{32}$$

or, from eq. (22),

$$q_1^2\,(i_1+c^2)\,(2x_1) \approx q_2^2\,(i_2+c^2)\left(2x_2-\frac{3}{x_2}+\frac{4\pi^2\varepsilon_2^2}{x_2}\right). \tag{33}$$

Solving for q in eq. (21), one has

$$q_i^2 = \frac{\Delta^2}{n_i^2+\Delta^2/c^2}\,. \tag{34}$$

Substituting q_i^2 in eq. (34) into eq. (33), one finds that

$$\frac{n_2^2+\frac{\Delta^2}{c^2}}{n_1^2+\frac{\Delta^2}{c^2}} \approx \frac{(i_2+c^2)\left(2x_2-\frac{3}{x_2}+\frac{4\pi^2\varepsilon_2^2}{x_2}\right)}{(i_1+c^2)\,(2x_1)} \tag{35}$$

Assuming that

$$i_1 + c^2 \approx c^2\,,$$

the right hand side of eq. (35) can be simplified as

$$\frac{1}{2x_1}\left[\frac{nl}{\mu H}(x_2 - 1 + \pi^2 \xi_2^2) + 1\right]\left[2x_2 - \frac{3}{x_2} + \frac{4\pi^2\xi_2^2}{x_2}\right],$$

or, by eq. (27),

$$\frac{1}{2x_1}\,\frac{n_2}{n_1}\left[2x_2 - \frac{3}{x_2} + \frac{4\mu H}{m\,x_2}\cdot\frac{n_2 - n_1}{n_1}\right].$$

Thus, eq. (35) becomes as follows:

$$\frac{n_2^2 + \frac{\Delta^2}{c^2}}{n_1^2 + \frac{\Delta^2}{c^2}} \approx \frac{n_2}{n_1}\,\frac{2\mu H}{m}\,\frac{1}{x_1 x_2}\,\frac{n_2 - n_1}{n} \tag{36}$$

Consider first the case where

i) $x_2 > x_1$;

Equation (36) appears to be true because both sides of the equation become larger than one and positive.

For the case where

ii) $x_2 < x_1$,

one finds that the left hand side is positive and less than one, while the right hand side of eq. (36) becomes negative, yielding a contradiction. More careful study reveals that when $n_1 = n_2$, eq. (36) may hold true. However, $n_1 = n_2$ implies the case where

$$q_1 = q_2$$

$$i_1 = i_2$$

$$P_1 = P_2$$

which is, of course, unphysical, so that it should be discarded. Thus, one is led to the condition

$x_2 - x_1 > 0$ for the physical processes.

These results imply that the passage of the shock through the medium increases the density, and greatly increases the temperature. The implications for advanced-stage thermonuclear reactions seem clear.

IV. CONCLUSION

It is found that the quantum nature of the neutral Fermi gas must be taken into account in considering the propagation of an acoustic wave through the gas. We find that a large heating effect is to be expected, which agrees in principle with current ideas relating to supernova explosions.

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APPENDIX

The Rankine-Hugoniot relations are known to be jump conditions for shocks, which can be derived by applying the three general principles -- conservation of mass, of momentum, and of energy -- to a column of gas in a tube. It seems appropriate to begin with Boltzmann's equation as a basic equation to describe the transport phenomena of degenerate gas.

Let us assume the column covers at time t the interval $a_0(t) < x < a_1(t)$, where $a_0(t)$ and $a_1(t)$ denote the positions of the moving particles that form the ends of the column, and the flow is supposed to be continuous at the ends of the column. Furthermore, we assume that in the moving column there is a point of discontinuity whose coordinate $x = \xi(t)$ moves with velocity $\dot{\xi}(t) = u(t)$.

Following Kadanoff and Baym(5), one has the following conservation laws for degenerate gas:

$$\frac{\partial}{\partial t}\langle n(x,t)\rangle + \underset{\sim}{\nabla}\cdot\langle \underset{\sim}{j}(x,t)\rangle = 0 \qquad \text{(mass)} \qquad (1)$$

$$\frac{\partial}{\partial t} m\langle \underset{\sim}{j}(x,t)\rangle + \underset{\sim}{\nabla}\cdot \underset{\approx}{J}(x,t) = -\langle n(x,t)\rangle \underset{\sim}{\nabla} U(x,t) \qquad \text{(momentum)} \qquad (2)$$

$$\frac{\partial}{\partial t}\varepsilon(x,t) + \underset{\sim}{\nabla}\cdot \underset{\sim}{j}_\varepsilon(x,t) = -\underset{\sim}{\nabla} U(x,t)\cdot\langle \underset{\sim}{j}(x,t)\rangle. \qquad \text{(energy)} \qquad (3)$$

where $\langle n(x,t)\rangle$ is the number density of particles or quasi-particles* averaged over the various momentum; that is

$$\langle n(x,t)\rangle \equiv \int d^3\underset{\sim}{p}\; n(\underset{\sim}{p}, x, t) \qquad (4)$$

and $\langle j(x,t)\rangle$ is the current density of particles averaged over momenta,

$$\langle \underset{\sim}{j}(x,t)\rangle = \int d^3\underset{\sim}{p}\; \underset{\sim}{\nabla}_p E(\underset{\sim}{p},x,t)\; n(\underset{\sim}{p},x,t). \qquad (5)$$

---

* We will use the terms particle or quasi-particle interchangeably, depending on whether the medium is a Fermi gas or a Fermi fluid.

with E(p,x,t) total energy of a quasi particle. Here m is a mass of the particle, U is the external potential, and $\underset{\approx}{J}(x,t)$ is the momentum stress tensor, expressible as follows:

$$J_{ij}(x,t) = \int d^3\underset{\sim}{p}\, n(\underset{\sim}{p},x,t)\left[p_j \frac{\partial E(\underset{\sim}{p},x,t)}{\partial p_i} + \delta_{ij} E(\underset{\sim}{p},x,t)\right] - \mathcal{E}(x,t)\,\delta_{ij} \tag{6}$$

With $\mathcal{E}(x,t)$, the energy density, given by

$$\mathcal{E}(x,t) = \int d^3\underset{\sim}{p}\, \frac{p^2}{2m} n(\underset{\sim}{p},x,t) + \frac{1}{2}\int d^3\underset{\sim}{p}\, d^3\underset{\sim}{p}'\, n(\underset{\sim}{p},x,t)\, n(\underset{\sim}{p}',x,t)\, f(\underset{\sim}{p},\underset{\sim}{p}';x,t) \tag{7}$$

The function $f(p,p';x,t)$ is defined by

$$f(p,p';x,t) \equiv \frac{\delta E(\underset{\sim}{p},x,t)}{\delta n(p',x,t)}(2\pi)^3, \tag{8}$$

which can be interpreted as the increment of particle energy at (p,x,t) due to the addition of a particle at (p',x,t). Finally

$$\underset{\sim}{j}_{\mathcal{E}}(x,t) = \int d^3\underset{\sim}{p}\, E(\underset{\sim}{p},x,t)\, \underset{\sim}{\nabla}_p E(\underset{\sim}{p},x,t)\, n(\underset{\sim}{p},x,t). \tag{9}$$

Considering the limiting process where the length of column approaches zero, one has generally equations of the form

$$\lim_{a_1-a_0\to 0} \int_{a_0}^{a_1} \left\{(\Psi + \underset{\sim}{\nabla}\cdot\underset{\sim}{\varphi}) + \Psi\,\Delta\eta\right\} dx = 0 \tag{10}$$

As long as $\Psi(x,t)$ does not behave like a delta-function, one has

$$\lim_{a_1-a_0\to 0} \int_{a_0}^{a_1} \Psi(x,t)\, dx \leq |\mathrm{Max}\, \Psi(x,t)|\cdot(a_1-a_0) \to 0, \tag{11}$$

while the second term in eq. (10) yields always

$$\underset{\sim}{\varphi}\Big|_{a_0}^{a_1} = \underset{\sim}{\varphi}(a_1) - \underset{\sim}{\varphi}(a_0).$$

The third term in eq. (10) yields

$$\lim_{a_1-a_0\to 0} \int_{a_0}^{a_1} (\Psi\,\Delta\eta)\, dx \leq \max|\Psi\,\Delta\eta|\cdot(a_1-a_0) \to 0, \tag{12}$$

provided also that $\Delta\eta$ does not behave like a delta function. Therefore, with the assumption

$$|\Psi|,\ |\Delta\eta| \approx \text{finite},\ \text{for}\ a_0 \leqslant x \leqslant a_1, \tag{13}$$

eq. (10) gives

$$\underset{\sim}{\varphi}(a_1) = \underset{\sim}{\varphi}(a_0) \tag{14}$$

This is the jump condition we seek.

With the above reasoning, one gets from eqs. (1), (2) and (3), the following relations, respectively:

$$\langle \underset{\sim}{j}(a_1,t)\rangle = \langle \underset{\sim}{j}(a_0,t)\rangle \tag{15}$$

$$\langle T_{ij}(a_1,t)\rangle = \langle T_{ij}(a_0,t)\rangle \tag{16}$$

$$\langle \underset{\sim}{j}_{\varepsilon}(a_1,t)\rangle = \langle \underset{\sim}{j}_{\varepsilon}(a_0,t)\rangle, \tag{17}$$

Here $a_1(t)$, $a_0(t)$ refer to the points front and rear of the discontinuous shock region. We may also derive from eq. (1), together with the second law of thermodynamics,

$$\langle \underset{\sim}{j}\, s\,(a_1,t)\rangle \geqslant \langle \underset{\sim}{j}\, s\,(a_0,t)\rangle \tag{18}$$

where s is the entropy per particle. Consequently, in order to obtain the Rankine-Hugoniot relations for a degenerate gas, one has to evaluate $\langle \underset{\sim}{j}(x,t)\rangle$, $\langle T_{ij}(x,t)\rangle$, $\langle \underset{\sim}{j}_{\varepsilon}(x,t)\rangle$ and $\langle \underset{\sim}{j}(x,t)s(x,t)\rangle$, explicitly under the specific conditions and interactions of the gas.

In the case of local equilibrium, one has

$$\underset{\sim}{j} = \underset{\sim}{v}(x,t)\ n(x,t)\ , \tag{19}$$

$$T_{ij} = mv_i(x,t)\ v_j(x,t)\ n + \delta_{ij}\ P(x,t) \tag{20}$$

and

$$\underset{\sim}{j}_{\varepsilon} = \underset{\sim}{v}\left\{\frac{m}{2} v^2 n + P(\underset{\sim}{x},t) + \varepsilon(\underset{\sim}{x},t)\right\} , \tag{21}$$

which obviously reduce to the conventional form of the Rankine-Hugoniot relations.

Thus, one is led to the conclusion that the Rankine-Hugoniot relations have the same structure for a degenerate gas as for classical continuous media, but that the terms should be expressed properly for the degenerate gas using the specific interaction between the particles.

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